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CONNECTION BETWEEN COMMUTATIVE ALGEBRA AND TOPOLOGY

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Abstract. The main aim of this paper to show how commutative algebra is connected to topology. We give underlying topological idea of some results on completable unimodular rows.

Keywords: Unimodular row, Matrix, Homotopy.

1. Introduction

Let k be a field and $\vec{x} = (x_1, x_2 \cdots, x_n) \in k^n$ be a non-zero vector, then $\vec{x}$ can be completed to a basis of k^n. We wish to have an analogue of the above statement for rings. Let A be a ring and $\vec{a} \in A^n, \vec{a} \neq 0$. Then there is a natural question when can $\vec{a}$ be completed to a basis of A^n? Suppose $\vec{a} = (a_1, a_2, \cdots, a_n)$ can be completed to a basis of A^n. Consider these basis vectors as columns of a matrix α, whose first column is $(a_1, a_2, \cdots, a_n)$. The fact that these vectors span A^n imply that there exists a $n \times n$ invertible matrix β such that $\alpha\beta = I_n$. Conversely if there exists an invertible matrix $\alpha \in M_n(A)$ with first column $(a_1, a_2, \cdots, a_n)$, then $(a_1, a_2, \cdots, a_n)$ can be completed to a basis of A^n. Since any completable row $(a_1, a_2, \cdots, a_n)$ is unimodular, this leads to the following problem : Suppose $(a_1, a_2, \cdots, a_n) \in A^n$ is a unimodular row. Then can one complete the row $(a_1, a_2, \cdots, a_n)$ to a matrix belonging to $GL_n(A)$?

In general answer of this question is negative. Surprisingly this is related to topology. Suppose one can find a matrix $\alpha \in GL_n(A)$ having first column $(a_1, a_2, \cdots, a_n)$. Then $e_1\alpha^t = (a_1, a_2, \cdots, a_n)$, where $e_1 = (1, 0, \cdots, 0)$. The group $GL_n(A)$ acts on A^n via matrix multiplication. The row $(a_1, a_2, \cdots, a_n)$ can be completed to a matrix in $GL_n(A)$ if and only if $(a_1, a_2, \cdots, a_n)$ lies in the orbit of $(1, 0, \cdots, 0)$ under the $GL_n(A)$ action (a similar statement holds for $SL_n(A)$).

Example 1.1. *Let $(x_1, x_2) \in \mathbb{Z}^2$. Then (x_1, x_2) is unimodular if and only if x_1, x_2 are relatively prime. In this case there exist $x, y \in \mathbb{Z}$ such that $x_1x + x_2y = 1$ and the matrix $\begin{pmatrix} x_1 & -y \\ x_2 & x \end{pmatrix}$ has determinant 1. We can find an explicit completion of (x_1, x_2) using the Euclidean Algorithm in the following manner: Assume for simplicity that $x_1, x_2 > 0$ and $x_1 > x_2$. Then by division algorithm $x_1 = x_2q + r$, where q is the quotient and r is the remainder. Then $\begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} r \\ x_2 \end{pmatrix}$. It follows by iterating the above procedure*

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that we can get a matrix α which is a product of matrices of the form $\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ q' & 1 \end{pmatrix}$ *where $q, q' \in \mathbb{Z}$ such that* $\alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. *Then* $\alpha^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ *and* $\alpha \in SL_2(\mathbb{Z})$.

This example motivates to define $E_n(A)$ (see 2.6.3).

Definition 1.2. *Let A be a commutative ring with identity. Let $e_{ij}(\lambda), i \neq j$ be the $n \times n$ matrix in $SL_n(A)$ which has 1 as its diagonal entries and λ as its $(i,j)^{th}$ entry. Let $E_n(A)$ be the subgroup of $SL_n(A)$ generated by $e_{ij}(\lambda), i \neq j$. We call elements of $E_n(A)$ as elementary matrices.*

For any Euclidean domain A, any unimodular row $(a_1, a_2, \cdots, a_n) \in A^n$ can be completed to an elementary matrix, where $n \geq 2$ *i.e.* $(a_1, a_2, \cdots, a_n) \overset{E_n(A)}{\sim} (1, 0, \cdots, 0)$, where $\overset{E_n(A)}{\sim}$ denotes the induced action of $E_n(A)$ on unimodular rows.

Note that $E_n(A) \subset SL_n(A) \subset GL_n(A)$, hence $(a_1, a_2, \cdots, a_n) \overset{E_n(A)}{\sim} (1, 0, \cdots, 0) \Rightarrow (a_1, a_2, \cdots, a_n) \overset{SL_n(A)}{\sim} (1, 0, \cdots, 0) \Rightarrow (a_1, a_2, \cdots, a_n) \overset{GL_n(A)}{\sim} (1, 0, \cdots, 0)$. Since $A = k[X]$ (where k is a field) is a Euclidean domain, $(a_1, a_2, \cdots, a_n) \overset{E_n(A)}{\sim} (1, 0, \cdots, 0)$ for any unimodular row $(a_1, a_2, \cdots, a_n) \in A^n$.

Proposition 1.3. *Let A be a ring and $(a_1, a_2, \cdots, a_n) \in A^n$ be a unimodular row. Then $(a_1 + \lambda X a_2, a_2, \cdots, a_n) \in A[X]^n$ is a unimodular row over $A[X]$, for every $\lambda \in A$.*

Proof. Since $(a_1, a_2, \cdots, a_n) \in A^n$ is a unimodular, there exists $(b_1, b_2, \cdots, b_n) \in A^n$ such that $\sum_{i=1}^n a_i b_i = 1$. Take $c_1 = b_1, c_2 = b_2 - b_1 \lambda X, c_3 = b_3, \cdots, c_n = b_n$, thus we have $c_1(a_1 + \lambda X a_2) + \sum_{i=2}^n a_i c_i = 1$. Therefore $(a_1 + \lambda X a_2, a_2, \cdots, a_n)$ is a unimodular row over $A[X]$. ☐

From Proposition 1.3, we can prove that for any matrix $\sigma = \prod_{i=1}^r E_{ij}(\lambda) \in E_n(A)$, $(a_1, a_2, \cdots, a_n)\sigma(X) \in A[X]^n$ is a unimodular over $A[X]$, where $\sigma(X) = \prod_{i=1}^r E_{ij}(\lambda X)$.

2. Topological fact

Lemma 2.1. *Let T be a topological space. Then any map $f : T \longrightarrow T$ is homotopic to itself.*

Lemma 2.2 ([6], Exercise 1, page 325)**.** *Let T, T_1 and T_2 be topological spaces. Suppose maps $h, h' : T \longrightarrow T_1$ and $k, k' : T_1 \longrightarrow T_2$ are homotopic. Then $k \circ h$ and $k' \circ h'$ are homotopic.*

Lemma 2.3 ([6])**.** *Let T, T_1 and T_2 be topological spaces. Suppose $k : T \longrightarrow T_1$ is a continuous map and F is a homotopy between maps $f, f' : T_2 \longrightarrow T$, then $k \circ F$ is a homotopy between the maps $k \circ f$ and $k \circ f'$.*

Definition 2.4. *Let $f : T \longrightarrow T'$ be a continuous map. We say that f is a homotopy equivalence if there exists a continuous map $g : T' \longrightarrow T$ such that $f \circ g$ is homotopic to the identity map $I_{T'}$ on T' and $g \circ f$ is homotopic to the identity map I_T on T. Two spaces T and T' are said to be homotopically equivalent or of the same homotopy type if there exists a homotopy equivalence from one to the other.*

Theorem 2.5 ([3])**.** *If two spaces T and T' are homotopically equivalent and any continuous map from T to S^{n-1} is homotopic to a constant map, then any continuous map from T' to S^{n-1} is homotopic to a constant map.*

Theorem 2.6 ([3])**.** *Let T be a simplicial complex of dimension $\leq r$ and T' be a closed subcomplex of T. Suppose $f : T' \longrightarrow S^r$ is a continuous map, Then f can be extended to a continuous map $f' : T \longrightarrow S^r$ such that $f'|T' = f$*

This topological fact can be proved by induction. One chooses a vertex of the complex T that does not belong to T' and chooses an arbitrary extension of f. Then one extends f linearly to the edge of T that does not belong to T', then to the two faces etc.

Theorem 2.7 ([3])**.** *Let T be a simplicial complex of dimension n and T' be a closed subcomplex of T. Then any continuous map $f : T' \longrightarrow \mathbb{R}^n - \{0\}$ can be extended to a continuous map $f' : T \longrightarrow \mathbb{R}^n$ such that $f'^{-1}(\{0\})$ is a finite set.*

Theorem 2.8 ([9], Theorem 4.4, page 153)**.** *Let T be a separable metric space and T' be a closed subspace of T. Suppose two maps f and g from $T' \longrightarrow S^n$ are homotopic. If there exists an extension $f' : T \longrightarrow S^n$ of f, then there also exists an extension $g' : T \longrightarrow S^n$ of g such that f' and g' are homotopic.*

Let I be an ideal of $k[X_1, X_2, \cdots, X_m]$. Then $V(I) = \{(x_1,\ x_2,\ \cdots,\ x_m) \in k^m \mid f(x_1, x_2, \cdots, x_m) = 0$, for every $f \in I\}$. By Hilbert basis theorem, every ideal of $k[X_1, X_2, \cdots, X_m]$ is finitely generated, so $V(I)$ is the set of common zeros of finitely many polynomials.

Example 2.9. *Let $A = \mathbb{R}[X_1, X_2]$ and $I = (X_1^2 + X_2^2 - 1) \subset A$ be an ideal , then $V(I) \cap \mathbb{R}^2 = S^1$ (real sphere).*

Since $V(I) \cap \mathbb{R}^m$ is the set of common zeros of finitely many polynomials, it is a closed set in the usual Euclidean topology in $\mathbb{R}^m$, where I be an ideal of $\mathbb{R}[X_1, X_2, \cdots, X_m]$. More generally for any field k, there exists a topology on k^m, where the subsets of the form $V(I)$ are closed. This topology is called the Zariski topology on k^m.

In topology Tietze extension theorem ([6], Theorem 3.2, page 212) says that "Any continuous map of a closed subset of a normal topological space T into the reals $\mathbb{R}$ may be extended to a continuous map of T into $\mathbb{R}$". As an algebraic analogue "any polynomial function on $V(I) \cap \mathbb{R}^m$ is the restriction of a polynomial function on $\mathbb{R}^m$".

Remark 2.1.
(1) Let $A = \mathbb{R}[X_1, X_2, \cdots, X_m]$, $T = Spec(A)$ and $V(0) = \mathbb{R}^m$. Let $\vec{a} = (a_1, a_2, \cdots, a_n) \in A^n$. Then we have a continuous map $F_{\vec{a}} : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ defined as $F_{\vec{a}}(x_1, x_2, \cdots, x_m) = (a_1(x_1, x_2, \cdots, x_m), \cdots, a_n(x_1, x_2, \cdots, x_m))$, for every $(x_1, x_2, \cdots, x_m) \in \mathbb{R}^m$. Similarly if $A = \mathbb{R}[X_1, X_2, \cdots, X_m]/I$, $T = Spec(A)$ and $V_I(\mathbb{R}) = V(I) \cap \mathbb{R}^m$, where I is an ideal of a real algebraic variety in $\mathbb{R}[X_1, X_2, \cdots, X_m]$. Then for $\vec{a} = (a_1, a_2, \cdots, a_n) \in A^n$, we have a continuous map $F_{\vec{a}} : V_I(\mathbb{R}) \longrightarrow \mathbb{R}^n$ defined as $F_{\vec{a}}(x_1, x_2, \cdots, x_m) = (a_1(x_1, x_2, \cdots, x_m),\ \cdots, a_n(x_1, x_2, \cdots, x_m))$, for every $(x_1, x_2, \cdots, x_m) \in V_I(\mathbb{R})$. The Hilbert Nullstellensatz says that if "$A = k[X_1, X_2, \cdots, X_m]$, where k is algebraically closed field and $a_1, a_2, \cdots, a_n \in A$, then $a_1, a_2, \cdots, a_n$ have a common zero in k^m if and only if the ideal $\langle a_1, a_2, \cdots, a_n\rangle \neq A$". Similar to the Hilbert Nullstellensatz if $A = \mathbb{R}[X_1, X_2, \cdots, X_m]/I$ and $\vec{a} = (a_1, a_2, \cdots, a_n) \in A^n$ is unimodular *i.e.* $\sum_{i=1}^n a_i b_i = 1$ for some $(b_1, b_2, \cdots, b_n) \in A^n$, then $a_1, a_2, \cdots, a_n$ do not simultaneously vanish at any point of $V_I(\mathbb{R})$. So we have a map $F_{\vec{a}} : V_I(\mathbb{R}) \longrightarrow \mathbb{R}^n - \{0, \cdots, 0\}$.

(2) Now define a map $g : \mathbb{R}^n - \{(0, 0, \cdots, 0)\} \longrightarrow S^{n-1}$ as $g(x) = \dfrac{x}{||x||}$ for every $x \in \mathbb{R}^n - \{(0, 0, \cdots, 0)\}$, where $||x||$ denotes norm of x. Thus we have a map $G_{\vec{a}} = g \circ F_{\vec{a}} : V_I(\mathbb{R}) \longrightarrow S^{n-1}(\subset \mathbb{R}^n - \{(0, 0, \cdots, 0)\})$. This shows that for any unimodular row $\vec{a} = (a_1, a_2, \cdots, a_n)$ over A, we have two continuous maps $F_{\vec{a}}$ and $G_{\vec{a}}$.
Claim: $F_{\vec{a}}$ and $G_{\vec{a}}$ are homotopic.
Since identity map Id on $\mathbb{R}^n - \{(0, 0, \cdots, 0)\}$ is homotopic to g (straight line homotopy) and $F_{\vec{a}}$ is homotopic to itself, map $Id \circ F_{\vec{a}} = F_{\vec{a}} : V_I(\mathbb{R}) \longrightarrow \mathbb{R}^n - \{(0, 0, \cdots, 0)\}$ is homotopic to $G_{\vec{a}} = g \circ F_{\vec{a}} : V_I(\mathbb{R}) \longrightarrow S^{n-1}$. This proves the claim.

(3) Let $A[X] = (\mathbb{R}[X_1, X_2, \cdots, X_m]/I)[X]$, where I is an ideal of a real algebraic variety in $\mathbb{R}[X_1, X_2, \cdots, X_m]$. Then any element $a \in I$ vanishes on $(x_1, x_2, \cdots, x_m, x)$ for

any $x \in \mathbb{R}$ and $(x_1, x_2, \cdots, x_m) \in V_I(\mathbb{R})$. This shows that the real points of the variety corresponding to $A[X]$ is $V_I(\mathbb{R}) \times \mathbb{R}$, where $V_I(\mathbb{R})$ is the set of real points of the variety corresponding to A. Therefore any unimodular row $(a_1(X), a_2(X), \cdots, a_n(X)) \in A[X]^n$ gives two maps $F_{\vec{a}}[X] : V_I(\mathbb{R}) \times \mathbb{R} \longrightarrow \mathbb{R}^n - \{0, \cdots, 0\}$ and $G_{\vec{a}}[X] : V_I(\mathbb{R}) \times \mathbb{R} \longrightarrow S^{n-1}$.

Throughout this chapter for any unimodular row $\vec{a} = (a_1, a_2, \cdots, a_n)$, $F_{\vec{a}}$ and $G_{\vec{a}}$ denote maps from $V_I(\mathbb{R}) \longrightarrow \mathbb{R}^n - \{(0, 0, \cdots, 0)\}$ and from $V_I(\mathbb{R}) \longrightarrow S^{n-1}$, respectively. Similarly for any unimodular row $\vec{a}[X] = (a_1(X), a_2(X), \cdots, a_n(X))$, $F_{\vec{a}}[X]$ and $G_{\vec{a}}[X]$ denote maps from $V_I(\mathbb{R}) \times \mathbb{R} \longrightarrow \mathbb{R}^n - \{(0, 0, \cdots, 0)\}$ and from $V_I(\mathbb{R}) \times \mathbb{R} \longrightarrow S^{n-1}$, respectively.

Proposition 2.10. *Let $\vec{a} = (a_1, a_2, \cdots, a_n)$ and $\vec{b} = (b_1, b_2, \cdots, b_n)$ be two unimodular rows over $A = \mathbb{R}[X_1, X_2, \cdots, X_m]/I$ such that $(a_1, a_2, \cdots, a_n) \overset{E_n(A)}{\sim} (b_1, b_2, \cdots, b_n)$, where I is an ideal of a real algebraic variety in $\mathbb{R}[X_1, X_2, \cdots, X_m]$. Then the corresponding mappings $F_{\vec{a}} : V_I(\mathbb{R}) \longrightarrow \mathbb{R}^n - \{(0, 0, \cdots, 0)\}$ and $F_{\vec{b}} : V_I(\mathbb{R}) \longrightarrow \mathbb{R}^n - \{(0, 0, \cdots, 0)\}$ are homotopic.*

Proof. Since $\vec{a} \overset{E_n(A)}{\sim} \vec{b}$, there exists $\sigma = \prod_{i=1}^{r} E_{ij}(\lambda) \in E_n(A)$ such that $\vec{a}\sigma = \vec{b}$. Take $\sigma(X) = \prod_{i=1}^{r} E_{ij}(\lambda X) E_n(A[X])$, so we have $\sigma(0) = I_n$ and $\sigma(1) = \sigma$. Thus $\vec{a}\sigma(0) = \vec{a}I_n = \vec{a}$ and $\vec{a}\sigma(1) = \vec{a}\sigma = \vec{b}$. From Proposition 1.3, $(a_1, a_2, \cdots, a_n)\sigma(X)$ is a unimodular row over $A[X]$, from Remark 2.1 (3), we have map $F_{\vec{a}}[X] : V_I(\mathbb{R}) \times \mathbb{R} \longrightarrow \mathbb{R}^n - \{(0, 0, \cdots, 0)\}$ such that $F_{\vec{a}}[0] = F_{\vec{a}}$ and $F_{\vec{a}}[1] = F_{\vec{b}}$. Thus the maps $F_{\vec{a}} : V_I(\mathbb{R}) \longrightarrow \mathbb{R}^n - \{(0, 0, \cdots, 0)\}$ and $F_{\vec{b}} : V_I(\mathbb{R}) \longrightarrow \mathbb{R}^n - \{(0, 0, \cdots, 0)\}$ are homotopic via $F_{\vec{a}}[X]$. $\square$

Note: From Remark 2.1 (2), maps $F_{\vec{a}}$ and $G_{\vec{a}}$ are homotopic and from Proposition 2.10, maps $F_{\vec{a}}$ and $F_{\vec{b}}$ are homotopic. Hence $G_{\vec{a}}$ and $G_{\vec{b}}$ are also homotopic (*i.e.* $G_{\vec{a}} \overset{homo}{\sim} F_{\vec{a}} \overset{homo}{\sim} F_{\vec{b}} \overset{homo}{\sim} G_{\vec{b}} \Longrightarrow G_{\vec{a}} \overset{homo}{\sim} G_{\vec{b}}$).

Now we will give underlying topological idea of some results on unimodular rows, which shows that how one can think of the following results from topological point of view.

Lemma 2.11. *Let $\vec{a} = (a_1, a_2, \cdots, a_n) \in Um_n(A)$ with a_1 being a unit of A. Then $(a_1, a_2, \cdots, a_n) \overset{E_n(A)}{\sim} (1, 0, \cdots, 0)$.*

Underlying topological idea: Suppose $A = \mathbb{R}[X_1, X_2, \cdots, X_m]/I$ and $V_I(\mathbb{R}) = V(I) \cap \mathbb{R}^m$. Since $(a_1, a_2, \cdots, a_n) \in A^n$ is a unimodular row over A, we have a map $G_{\vec{a}} : V_I(\mathbb{R}) \longrightarrow S^{n-1}$. Since a_1 is a unit, $a_1 b_1 = 1$ for some $b_1 \in A$ *i.e.* a_1 does not vanish at any point of $V_I(\mathbb{R})$. Therefore for any element of $\vec{p} = (0, x_2, \cdots, x_n) \in S^{n-1}$, there does not exist an element $\vec{x} \in V_I(\mathbb{R})$ such that $G_{\vec{a}}(\vec{x}) = \vec{p}$. In other words $G_{\vec{a}}$ is not surjective. So $\mathrm{Im}(G_{\vec{a}}) \subset S^{n-1} - \{\vec{p}\}$. Since $S^{n-1} - \{\vec{p}\}$ is contractible, map $G_{\vec{a}}$ is homotopic to a constant map.

Lemma 2.12. *Let $\vec{a} = (a_1, a_2, \cdots, a_n) \in A^n$ be a unimodular row. Suppose $(a_1, a_2, \cdots, a_i)$, for any $i < n$, is unimodular. Then $(a_1, a_2, \cdots, a_n) \overset{E_n(A)}{\sim} (1, 0, \cdots, 0)$.*

Underlying topological idea: Suppose $A = \mathbb{R}[X_1, X_2, \cdots, X_m]/I$ and $V_I(\mathbb{R}) = V(I) \cap \mathbb{R}^m$. Since $(a_1, a_2, \cdots, a_n) \in A^n$ is a unimodular row over A, we have a map $G_{\vec{a}} : V_I(\mathbb{R}) \longrightarrow S^{n-1}$. Since $(a_1, a_2, \cdots, a_i)$ is a unimodular, $a_1, a_2, \cdots, a_i$ do not vanish simultaneously at any point of $V_I(\mathbb{R})$. Therefore for any element $\vec{q} \in S^{n-1}$, whose first i-th coordinates are zero, there does not exist an element $\vec{x} \in V_I(\mathbb{R})$ such that $G_{\vec{a}}(\vec{x}) = \vec{q}$. Hence proof is similar to the proof of Lemma 2.11.

The inclusion map from S^1 to $\mathbb{R}^2 - \{0\}$ is not homotopic to a constant map. As an algebraic consequence we have the following:-

Example 2.13. *The unimodular row $\vec{a} = (\boldsymbol{x}_1, \boldsymbol{x}_2) \in A^2$, where $A = \mathbb{R}[X_1, X_2]/(X_1^2 + X_2^2 - 1)$ satisfies the property that $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ can not be transformed to $(1, 0)$ via an element of $E_2(A)$ i.e. there does not exist a matrix $\sigma \in E_2(A)$ such that $(\boldsymbol{x}_1, \boldsymbol{x}_2)\sigma = (1, 0)$.*

Proof. Assume contrary. Since $(\mathbf{x}_1, \mathbf{x}_2) \in Um_2(A)$ and $V_{(X_1^2+X_2^2-1)}(\mathbb{R}) = V((X_1^2 + X_2^2 - 1)) \cap \mathbb{R}^2 = S^1$, we have a map $F_{\vec{a}} : S^1 \longrightarrow \mathbb{R}^2 - \{0\}$ which is a inclusion map. Suppose there exists $\sigma \in E_2(A)$ such that $(\mathbf{x}_1, \mathbf{x}_2)\sigma = (1, 0)$. Then it follows $(\mathbf{x}_1, \mathbf{x}_2)\sigma(X) \in Um_2(A[X])$ (1.3). So we have a map $F_{\vec{a}}[X] : S^1 \times \mathbb{R} \longrightarrow \mathbb{R}^2 - \{0\}$ such that $F_{\vec{a}}[0] = F_{\vec{a}}$ and $F_{\vec{a}}[1] =$ constant map. This shows that inclusion map $F_{\vec{a}}$ is homotopic to a constant map, which is not possible. Hence our assumption is not true. $\square$

Theorem 2.14 ([1], Theorem 9.3)**.** *Let A be a Noetherian ring of dimension d. Let $(a_1, a_2, \cdots, a_n) \in A^n$ be a unimodular row with $n \geq d + 2$. Then $(a_1, a_2, \cdots, a_n) \overset{E_n(A)}{\sim} (1, 0, \cdots, 0)$.*

Underlying topological idea: Let A be the coordinate ring of a real algebraic variety of dimension d. Then the dimension of $V_I(\mathbb{R}) \leq d$. Since $(a_1, a_2, \cdots, a_n)$ is unimodular, we have a continuous map $G_{\vec{a}} : V_I(\mathbb{R}) \longrightarrow S^{n-1}$. Assume that $V_I(\mathbb{R})$ is a simplicial complex. By simplicial approximation there exists a simplicial map $\psi : V_I(\mathbb{R}) \longrightarrow S^{n-1}$ such that ψ and $G_{\vec{a}}$ are homotopic via straight line homotopy. Since $n \geq d + 2$, $n - 1 \geq d + 1$. Therefore ψ is not surjective (because simplicial map can not raise dimension) *i.e.* $\mathrm{Im}(\psi) \subset S^{n-1} - \{\vec{p}\}$. Since $S^{n-1} - \{\vec{p}\}$ is contractible, map ψ is homotopic to a constant map. Hence $G_{\vec{a}}$ is also homotopic to a constant map.

Theorem 2.15. *Let A be a Noetherian ring of dimension d. Let $(a_1(X), a_2(X), \cdots, a_n(X)) \in A[X]^n$ be a unimodular row with $n \geq d + 2$. Then*

$$(a_1(X), a_2(X), \cdots, a_n(X)) \overset{E_n(A[X])}{\sim} (1, 0, \cdots, 0).$$

Underlying topological idea: Let A be the co-ordinate ring of a real algebraic variety of dimension d. Then the dimension of $V_I(\mathbb{R}) \leq d$. Assume that $V_I(\mathbb{R})$ is a simplicial complex. For $n \geq d + 3$, theorem follows from Theorem 2.14. Since $(a_1(X), a_2(X), \cdots, a_n(X)) \in A[X]^n$ is unimodular, we have a continuous map $G_{\vec{a}}[X] : V_I(\mathbb{R}) \times \mathbb{R} \longrightarrow S^{n-1}$. Consider inclusion map $i : V_I(\mathbb{R}) \longrightarrow V_I(\mathbb{R}) \times \mathbb{R}$ and projection $p : V_I(\mathbb{R}) \times \mathbb{R} \longrightarrow V_I(\mathbb{R})$. Then $p \circ i = Id_{V_I(\mathbb{R})}$ which is obviously homotopic to identity map on $V_I(\mathbb{R})$. On the other hand map $H(x, t) = t(i \circ p)(x) + (1 - t)Id_{V_I(\mathbb{R})\times\mathbb{R}}$ gives a homotopy between $i \circ p$ and $Id_{V_I(\mathbb{R})\times\mathbb{R}}$.

This shows that spaces $V_I(\mathbb{R})$ and $V_I(\mathbb{R}) \times \mathbb{R}$ are homotopically equivalent. Also from Theorem 2.14, any map from $V_I(\mathbb{R}) \longrightarrow S^{n-1}$ is homotopic to a constant map. Therefore from Theorem 2.5, $G_{\vec{a}}[X]$ is also homotopic to a constant map.

The following theorem is a particular case of the Lemma 4.2.13 (Chapter 4).

Theorem 2.16. *Let A be a Noetherian ring of dimension $\leq r$ and J be an ideal of A. Let $(\overline{a_1}, \overline{a_2}, \cdots, \overline{a_{r+1}}) \in (A/J)^{r+1}$ be a unimodular row. Then there exist $c_1, c_2, \cdots, c_{r+1}$ such that $\overline{c_i} = \overline{a_i}$ and $(c_1, c_2, \cdots, c_{r+1})$ is unimodular over A.*

Underlying topological idea: Suppose $A = \mathbb{R}[X_1, X_2, \cdots, X_m]/I$ and $V_I(\mathbb{R}) = V(I) \cap \mathbb{R}^m$. Assume that $V_I(\mathbb{R})$ is a simplicial complex. Take $V_J(\mathbb{R}) = V(J) \cap \mathbb{R}^m$. Then $V_J(\mathbb{R})$ is a closed subspace of $V_I(\mathbb{R})$. Since $(\overline{a_1}, \overline{a_2}, \cdots, \overline{a_{r+1}}) \in (A/J)^{r+1}$ is a unimodular row, we have a continuous map $G_{\vec{a}} : V_J(\mathbb{R}) \longrightarrow S^r$. By Theorem 2.6, $G_{\vec{a}}$ can be extended to $G'_{\vec{a}} : V_I(\mathbb{R}) \longrightarrow S^r$ such that $G'_{\vec{a}|V_J(\mathbb{R})} = G_{\vec{a}}$.

Now we will give motivation about algebraic proof of Theorem 2.16 by topological proof of Theorem 2.6.

Consider Spec(A) as a simplicial complex with vertices as minimal prime ideals of A, edges as height one prime ideals of A and triangles as height 2 prime ideals etc. We say that an element $a \in A$ vanishes on an edge corresponding to $\mathfrak{p}$ if $a \in \mathfrak{p}$. Also assume $J = \langle a_{r+2} \rangle$.

Now $K = \{\mathfrak{p} \in Spec(A) \mid a_{r+2} \in \mathfrak{p}\}$ is a sub-complex of Spec(A) and we have a row $(a_1, a_2, \cdots, a_{r+1})$ which does not vanish on K.

Now we choose vertices of Spec(A) which do not belong to K *i.e.* choose minimal prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_s$ of A such that $a_{r+2} \notin \mathfrak{p}_i$, $1 \leq i \leq s$. Thus $\langle a_1, a_{r+2} \rangle \notin \mathfrak{p}_i$, $1 \leq i \leq s$.

Therefore there exists $\lambda_1 \in A$ such that $a'_1 = a_1 + \lambda_1 a_{r+2} \notin \cup_{i=1}^{s} \mathfrak{p}_i$. Hence we have a row $(a'_1, a_2, \cdots, a_{r+1})$ which does not vanish on K', where $K' = K \cup (\cup_{i=1}^{s} \mathfrak{p}_i)$.

Now we choose edges of Spec(A) which do not belong to K' *i.e.* choose prime ideals $\mathfrak{q}_1, \mathfrak{q}_2, \cdots, \mathfrak{q}_l$ containing a'_1 but $a_{r+2} \notin \mathfrak{q}_i$. If no such prime ideal with the above property exists, then $(a'_1, a_2, \cdots, a_{r+1})$ does not vanish on Spec(A). So we are done. Otherwise, since $a_{r+2} \notin \mathfrak{q}_i$, $\langle a_2, a_{r+2} \rangle \nsubseteq \mathfrak{q}_i$, $1 \leq i \leq l \Rightarrow \langle a_2, a_{r+2} \rangle \nsubseteq \cup_{i=1}^{l} \mathfrak{q}_i$. So there exists λ_2 such that $a'_2 = a_2 + \lambda_2 a_{r+2} \notin \cup_{i=1}^{l} \mathfrak{q}_i$. Hence we have a row $(a'_1, a'_2, \cdots, a_{r+1})$ which does not vanish on K'', where $K'' = K' \cup (\cup_{i=1}^{l} \mathfrak{q}_i)$. Continuing same procedure we get the results.

The following theorem is a particular case of the Lemma 4.2.5 (Chapter 4).

Theorem 2.17. *Let A be a Noetherian ring of dimension n and J be an ideal of A. Suppose $(\overline{a_1}, \overline{a_2}, \cdots, \overline{a_n}) \in (A/J)^n$ is a unimodular row. Then there exist $b_1, b_2, \cdots, b_n \in A$ such that $\overline{b_i} = \overline{a_i}$, for all i and ideal $\langle b_1, b_2, \cdots, b_n \rangle$ has height n.*

Underlying topological idea: Suppose $A = \mathbb{R}[X_1, X_2, \cdots, X_m]/I$ and $V_I(\mathbb{R}) = V(I) \cap \mathbb{R}^m$. Assume that $V_I(\mathbb{R})$ is a simplicial complex. Take $V_J(\mathbb{R}) = V(J) \cap \mathbb{R}^m$. Then $V_J(\mathbb{R})$ is a closed subspace of $V_J(\mathbb{R})$. Since $(\overline{a_1}, \overline{a_2}, \cdots, \overline{a_{r+1}}) \in (A/J)^{r+1}$ is a unimodular row, we have a continuous map $G_{\vec{a}} : V_J(\mathbb{R}) \longrightarrow S^r$. By Theorem 2.7, $G_{\vec{a}}$ can be extended to a continuous map $G'_{\vec{a}} : V_I(\mathbb{R}) \longrightarrow S^r$ such that $G'_{\vec{a}|V_J(\mathbb{R})} = G_{\vec{a}}$.

Lemma 2.18. *The canonical homomorphism of groups from $E_n(A)$ to $E_n(A/I)$ is surjective.*

The lemma follows from the fact that generators $E_{ij}(\overline{\lambda})$ of $E_n(A/I)$ can be lifted to generators $E_{ij}(\lambda)$ of $E_n(A)$.

We show by an example that the canonical homomorphism from $SL_n(A)$ to $SL_n(A/I)$ need not be surjective.

Example 2.19. *Let $B = \mathbb{R}[X, Y], A = \mathbb{R}[X, Y]/(X^2 + Y^2 - 1)$. Then the unimodular row $(\boldsymbol{x}, \boldsymbol{y}) \in A^2$ can not be lifted to a unimodular row over B^2.*

Proof. Let $\alpha = \begin{pmatrix} \mathbf{x} & \mathbf{y} \\ -\mathbf{y} & \mathbf{x} \end{pmatrix} \in SL_2(A)$.
Claim: There does not exist $\beta \in SL_2(B)$ such that $\overline{\beta} = \alpha$.
Since $\alpha \in A$, we have a map $\phi : S^1 \longrightarrow SL_2(\mathbb{R})$ defined by

$$\phi(x_1, x_2) = \alpha(x_1, x_2) = \begin{pmatrix} \mathbf{x}(x_1, x_2) & \mathbf{y}(x_1, x_2) \\ -\mathbf{y}(x_1, x_2) & \mathbf{x}(x_1, x_2) \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix}.$$

Let

$$\beta = \begin{pmatrix} f_1(X, Y) & f_2(X, Y) \\ -f_3(X, Y) & f_4(X, Y) \end{pmatrix} \in SL_2(B)$$

be a lift of α. Therefore we have a map $\Phi : \mathbb{R}^2 \longrightarrow SL_2(\mathbb{R})$ defined by

$$\Phi(x_1, x_2) = \beta(x_1, x_2) = \begin{pmatrix} f_1(x_1, x_2) & f_2(x_1, x_2) \\ -f_3(x_1, x_2) & f_4(x_1, x_2) \end{pmatrix}$$

which is clearly an extension of ϕ. In particular considering the first row of α & β we see that inclusion map from $S^1 \longrightarrow \mathbb{R}^2 - \{0\}$ extends to $\mathbb{R}^2 \longrightarrow \mathbb{R}^2 - \{0\}$ which is not possible. Hence claim is proved. Thus the unimodular row $(\mathbf{x}, \mathbf{y}) \in A^2$ can not be lifted to a unimodular row over B^2.
Note: From Example 2.19, it is clear that $\begin{pmatrix} \mathbf{x} & \mathbf{y} \\ -\mathbf{y} & \mathbf{x} \end{pmatrix}$ does not belong to $E_2(A)$ otherwise it could be lifted to a matrix in $E_2(B) \subset SL_2(B)$ *i.e.* the unimodular row $(\mathbf{x}, \mathbf{y}) \in A^2$ is not elementary completable. □

The general from of the Lemma 2.18 is the following fact-

Theorem 2.20. *Let $\vec{a} = (a_1, a_2, \cdots, a_n) \in A^n$ be a unimodular row. Suppose J is an ideal of A and $(\overline{a_1}, \overline{a_2}, \cdots, \overline{a_n}) \overset{E_n(\overline{A})}{\sim} (\overline{b_1}, \overline{b_2}, \cdots, \overline{b_n})$, where bar denotes reduction modulo J. Then there exists $(c_1, c_2, \cdots, c_n) \in A^n$ such that $(a_1, a_2, \cdots, a_n) \overset{E_n(A)}{\sim} (c_1, c_2, \cdots, c_n)$ and $(\overline{c_1}, \overline{c_2}, \cdots, \overline{c_n}) = (\overline{b_1}, \overline{b_2}, \cdots, \overline{b_n})$. In particular $(\overline{b_1}, \overline{b_2}, \cdots, \overline{b_n})$ can be lifted to a unimodular row over A.*

Underlying topological idea: Suppose $A = \mathbb{R}[X_1, X_2, \cdots, X_m]/I$ and $V_I(\mathbb{R}) = V(I) \cap \mathbb{R}^m$. Assume that $V_I(\mathbb{R})$ is a separable metric space. Take $V_J(\mathbb{R}) = V(J) \cap \mathbb{R}^m$. Then $V_J(\mathbb{R})$ is a closed subspace of $V_I(\mathbb{R})$. Since $(\overline{a_1}, \overline{a_2}, \cdots, \overline{a_n}) \overset{\overline{\alpha} \in E_n(\overline{A})}{\sim} (\overline{b_1}, \overline{b_2}, \cdots, \overline{b_n})$, by Proposition 2.10, the corresponding continuous maps $G'_{\vec{a}} : V_J(\mathbb{R}) \longrightarrow S^{n-1}$ and $G'_{\vec{b}} : V_J(\mathbb{R}) \longrightarrow S^{n-1}$ are homotopic. That is there exists a continuous map $F : V_J(\mathbb{R}) \times \mathcal{I} \longrightarrow S^{n-1}$ such that $F(x, 0) = G'_{\vec{a}}$ and $F(x, 1) = G'_{\vec{b}}$.

Let α be the lift of $\overline{\alpha}$. Take $(c_1, c_2, \cdots, c_n) = (a_1, a_2, \cdots, a_n)\alpha$. Thus we have an extension $G_{\vec{a}} : V_I(\mathbb{R}) \longrightarrow S^{n-1}$ given by $(c_1, c_2, \cdots, c_n)$ of $G'_{\vec{a}}$.
Define $H' : (V_J(\mathbb{R}) \times \mathcal{I}) \cup (V_I(\mathbb{R}) \times \{0\}) \longrightarrow S^n$ by

$$H'(x,t) = \begin{cases} F(x,t) & \text{for} \quad x \in V_J(\mathbb{R}) \ \& \ 0 \le t \le 1 \\ G_{\vec{a}} & \text{for} \quad x \in V_I(\mathbb{R}) \ \& \ t = 0 \end{cases}$$

which is obviously a continuous map. Since $(V_J(\mathbb{R}) \times \mathcal{I}) \cup (V_I(\mathbb{R}) \times \{0\})$ is a closed subset of $V_I(\mathbb{R}) \times \mathcal{I}$, H' can be extended to $H : V_I(\mathbb{R}) \times \mathcal{I} \longrightarrow S^n$ (from proof of Theorem 2.8). Take $G_{\vec{b}} = H(x, 1) : V_I(\mathbb{R}) \longrightarrow S^n$. Then $G_{\vec{b}|V_J(\mathbb{R})} = G'_{\vec{b}}$.

Example 2.21. *Let $A = \mathbb{R}[X, Y, Z]/(X^2+Y^2+Z^2-1)$. Then $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in A^3$ is a unimodular row. Since the identity map from S^2 to itself is not homotopic to a constant map, we have as an algebraic consequence that (x, y, z) is not equivalent to $(1, 0, 0)$ via the action of $E_3(A)$. In other words $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is not completable to an elementary matrix. In fact $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is not completable to a matrix in $GL_3(A)$.*

Proof. Assume contrary that $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is the first row of a matrix in $GL_3(A)$ *i.e.* $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is a completable unimodular row. In other words $P \cong A^3/\langle \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle \cong A^2$. Thus we have a surjective homomorphism $f : P \longrightarrow A$. Suppose e_1, e_2, e_3 are the standard basis vectors of A^3 and $f(\overline{e_i}) = h_i$ for $i = 1, 2, 3$. Since $f(\mathbf{x}, \mathbf{y}, \mathbf{z}) = 0$, we have $\mathbf{x}f(\overline{e_1}) + \mathbf{y}f(\overline{e_2}) + \mathbf{z}f(\overline{e_3}) = 0 \Rightarrow \mathbf{x}h_1 + \mathbf{y}h_2 + \mathbf{z}h_3 = 0$. This implies that $Xh_1(X, Y, Z) + Yh_2(X, Y, Z) + Zh_3(X, Y, Z)$ is a multiple of $X^2 + Y^2 + Z^2 - 1$.

In particular, if $x_1^2+x_2^2+x_3^2 = 1$, then $x_1h_1(x_1, x_2, x_3)+x_2h_2(x_1, x_2, x_3)+x_3h_3(x_1, x_2, x_3) = 0$ implies that $(h_1(x_1, x_2, x_3), h_2(x_1, x_2, x_3), h_3(x_1, x_2, x_3))$ is perpendicular to (x_1, x_2, x_3). Define a continuous vector field $\Phi_1 : S^2 \longrightarrow \mathbb{R}^3$ by $\Phi_1(x_1, x_2, x_3) = (h_1(x_1, x_2, x_3), h_2(x_1, x_2, x_3), h_3(x_1, x_2, x_3))$. The zeros of this vector field are those point $(x_1, x_2, x_3) \in S^2$ where $h_1(x_1, x_2, x_3)$, $h_2(x_1, x_2, x_3)$, $h_3(x_1, x_2, x_3)$ are all zero. Since $f(\overline{e_i}) = h_i$, the ideal $(h_1(\mathbf{x}, \mathbf{y}, \mathbf{z}), h_2(\mathbf{x}, \mathbf{y}, \mathbf{z}), h_3(\mathbf{x}, \mathbf{y}, \mathbf{z})) = A$. Hence the corresponding vector field on S^2 has no real zeros, contradicting the fact that there is no nowhere vanishing continuous vector field on S^2.

Thus $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is not completable to a matrix in $GL_3(A)$ implies that $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is not completable to a matrix in $E_3(A)$. □

Proposition 2.22 ([**?**], Corollary 2.2)**.** *Let $(a, b, c) \in A^3$ be a unimodular row. Then (a^2, b, c) is completable.*

To understood underlying topological idea, we first give a proof of Proposition 2.22.

Proof. Since (a,b,c) is a unimodular, there exist a', b' & c' such that $aa'+bb'+cc'=1$. Consider the matrix

$$\alpha = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & c' & -b' \\ -b & -c' & 0 & a' \\ -c & b' & -a' & 0 \end{pmatrix}.$$

Then $det(\alpha) = (aa'+bb'+cc')^2 = 1$. Since $\langle 0,-a,-b,-c\rangle = A$, $(0,-a,-b,-c)$ is a elementary completable. The matrix $\alpha_1\alpha_2$ is a completion of $(0,-a,-b,-c)$, where

$$\alpha_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ c & 0 & 0 & 1 \end{pmatrix}$$

and

$$\alpha_2 = \begin{pmatrix} 1 & -a' & -b' & -c' \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Take $\alpha' = \alpha_1\alpha_2\alpha$. Then $\alpha' = \begin{pmatrix} 1 & a & b & c \\ 0 & a^2 & ab+c' & ac-b' \\ 0 & ab-c' & b^2 & bc+a' \\ 0 & ac+b' & bc-a' & c^2 \end{pmatrix}$

and $det(\alpha') = det(\alpha)$. By replacing $a \to a, b \to b, c \to c, a' \to a', b' \to b'+ac, c' \to c'-ab$, we have

$$\beta = \begin{pmatrix} 1 & a & b & c \\ 0 & a^2 & c' & -b' \\ 0 & 2ab-c' & b^2 & bc+a' \\ 0 & 2ac+b' & bc-a' & c^2 \end{pmatrix}.$$

Then $det(\beta) = aa' + b(b'+ac) + c(c'-ab) = aa'+bb'+cc' = 1$. Again by replacing $a \to a, b \to -b', c \to c', a' \to a', b' \to -b, c' \to c$, we have

$$\sigma = \begin{pmatrix} 1 & a & -b' & c' \\ 0 & a^2 & c & b \\ 0 & -2ab'-c & b'^2 & -b'c'+a' \\ 0 & 2ac'-b & -b'c'-a' & c'^2 \end{pmatrix}.$$

Then $det(\sigma) = aa' + (-b')(-b+ac') + c'(c+ab') = aa'+bb'+cc' = 1$. It is clear that $det(\sigma) = det\begin{pmatrix} a^2 & c & b \\ -2ab'-c & b'^2 & -b'c'+a' \\ 2ac'-b & -b'c'-a' & c'^2 \end{pmatrix} = 1$. Hence the matrix

$$\alpha_3 = \begin{pmatrix} a^2 & b & c \\ -2ac'-b & c'^2 & -b'c'+a' \\ 2ab'-c & -b'c'-a' & b'^2 \end{pmatrix}$$

is a completion of (a^2,b,c). $\square$

Underlying topological idea: Let $A = \mathbb{R}[X_1, X_2, \cdots, X_m]/I$ and $V_I(\mathbb{R}) = V(I) \cap \mathbb{R}^m$. Since $(a,b,c) \in A^3$ is a unimodular row, $(0,-a,-b,-c)$ is elementary completable.

Consider an exact sequence $0 \longrightarrow SL_3(\mathbb{R}) \xrightarrow{i_1} SL_4(\mathbb{R}) \xrightarrow{i_2} \mathbb{R}^4 - \{0\} \longrightarrow 0$, where i_1 sends any matrix $\sigma' \in SL_3(\mathbb{R})$ to $\begin{pmatrix} 1 & 0 \\ 0 & \sigma' \end{pmatrix} \in SL_4(\mathbb{R})$ and i_2 sends any matrix $\sigma_1 \in SL_4(\mathbb{R})$ to its first row.

Since $\alpha \in SL_4(A)$, we have a map $\phi_1 : V_I(\mathbb{R}) \longrightarrow SL_4(\mathbb{R})$ defined as $\phi_1(\vec{x}) = \alpha(\vec{x})$ and for any unimodular row of length 4, we have a map from $V_I(\mathbb{R}) \longrightarrow \mathbb{R}^4 - \{0\}$. The map

$i_2 \circ \phi_1 : V_I(\mathbb{R}) \longrightarrow \mathbb{R}^4 - \{0\}$ is equal to the map given by unimodular row $(0, -a, -b, -c)$ which is homotopic to constant map (by Proposition 2.10). Also $\alpha_3 \in SL_3(A)$ gives a map $\phi_2 : V_I(\mathbb{R}) \longrightarrow SL_3(\mathbb{R})$ such that $i_1 \circ \phi_2$ homotopic to the map ϕ_1 and the map $i_2' \circ \phi_2 : V_I(\mathbb{R}) \longrightarrow \mathbb{R}^3 - \{0\}$ is equal to the map given by unimodular row (a^2, b, c), where i_2' sends any matrix $\sigma_1 \in SL_3(\mathbb{R})$ to its first row.

3. On a lemma of Vaserstein's

Throughout this section T is a compact Hausdorff topological space (*i.e.* normal space), $C(T)$ is the ring of real valued continuous functions on T and $v_0 \overset{GL_n(C(T))}{\sim} v_t$ means there exists a matrix $\alpha \in GL_n(C(T))$ such that $v_0\alpha = v_t$, where v_0 and v_t are unimodular row in $C(T)$. This is an equivalence relation.

We now give a proof of a result which says that if there is no nowhere vanishing continuous vector field on S^2, then S^2 is not contractible. This proof is motivated from Simha ([7]).

To begin proof we need some preliminaries on reflections:-

Let $w(\neq 0) \in \mathbb{R}^n$ be a vector. A reflection about w is a linear transformation $\sigma : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ which satisfies $\sigma(w) = -w$, & $\sigma(w_1) = w_1$, where $w_1 \in w^{\perp} = \{w' \in \mathbb{R}^n \mid \langle w', w\rangle = 0\}$.

Let $v \in \mathbb{R}^n$ such that $v = v_1 + \lambda w, v_1 \in w^{\perp}$. Then $\sigma(v) = v_1 - \lambda w$. We have $\langle v, w\rangle = \langle v_1, w\rangle + \lambda\langle w, w\rangle = \lambda\langle w, w\rangle$, since $v_1 \in w^{\perp}$. This implies that $\lambda = \dfrac{\langle v, w\rangle}{\langle w, w\rangle}$. Therefore $\sigma(v) = v_1 - \lambda w = v_1 + \lambda w - 2\lambda w = v - 2\lambda w = v - 2\dfrac{\langle v, w\rangle}{\langle w, w\rangle}w$. This map σ is denoted by σ_w.

If v_1 and w_1 are two vectors in $\mathbb{R}^n$ and $||v_1|| = ||w_1||$, then we have a rhombus whose sides are v_1, w_1 and whose diagonal are $v_1 + w_1, v_1 - w_1$. Thus $\sigma_{v_1 - w_1}(v_1) = v_1 - 2\dfrac{\langle v_1, v_1 - w_1\rangle}{\langle v_1 - w_1, v_1 - w_1\rangle}(v_1 - w_1) = v_1 - 2\dfrac{||v_1||^2 - \langle v_1, w_1\rangle}{||v_1||^2 - 2\langle v_1, w_1\rangle + ||w_1||^2}(v_1 - w_1) = v_1 - (v_1 - w_1) = w_1$. Also we have $\sigma_{v_1 + w_1}(v_1) = -w_1$.

Remark 3.1. Any continuous map from $T \longrightarrow \mathbb{R}^n - \{0\}$ leads to a unimodular row $(a_1, a_2, \cdots, a_n)$ over the ring of continuous function $C(T)$, where a_i is the projection from $T \longrightarrow \mathbb{R}$ because the element $\sum a_i^2 \in \langle a_1, a_2, \cdots, a_n\rangle$ is a continuous function on T which does not vanish at any point of T otherwise each a_i will vanish at that point. In particular, any continuous map $T \longrightarrow S^{n-1}$ gives rise to a unimodular row over $C(T)$. Let $H : T \times \mathcal{I} \longrightarrow S^{n-1}$ be a continuous map.

Claim: For sufficiently small t, $v_0 \overset{GL_n(C(T))}{\sim} v_t$, where v_0 and v_t are the corresponding unimodular rows given by the maps $H(x, 0) : T \longrightarrow S^{n-1}$ and $H(x, t) : T \longrightarrow S^{n-1}$, respectively.

Proof. By continuity of H and compactness of T, it follows that there exists a $t_\theta > 0$ such that for $t < t_\theta$, v_0 and v_t are sufficiently close in the sense that $v_0(p)$ and $v_t(p)$ are not antipodal for every p *i.e.* $v_0(p) + v_t(p) \neq 0$ for $t < t_\theta$ and for every $p \in T$. Then for every $p \in T, \sigma_{v_0(p)+v_t(p)}$ is a reflection which is an element of $O_n(\mathbb{R})$ and satisfies $\sigma_{v_0(p)+v_t(P)}(v_0(p)) = -v_t(p)$. Hence there exists $\alpha : T \longrightarrow O_n(\mathbb{R})$ sending p to $-\sigma_{v_0(p)+v_t(p)}$ such that $\alpha v_0 = v_t$. Since $O_n(\mathbb{R}) \subset GL_n(\mathbb{R}), \sigma \in GL_n(C(T))$ and $v_0 \overset{GL_n(C(T))}{\sim} v_t$. □

Now we will give a proof of the fact that S^2 is not contractible.

Proof. Assume contrary that the real two sphere S^2 is contractible. Then there exists a continuous map $H : S^2 \times \mathcal{I} \longrightarrow S^2$, where $\mathcal{I} = [0, 1]$ such that the map $H(x, 0)$ is a constant map given by $(1, 0, 0)$ and $H(x, 1)$ is a identity map on S^2 corresponding to the unimodular row $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in (C(S^2))^3$.

Now let $S = \{t \in \mathcal{I} \mid v_0 \overset{GL_n(C(T))}{\sim} v_t\}$. By the compactness and connectedness of $\mathcal{I}$, it is easy to see that $S = \mathcal{I}$ *i.e.* $v_0 \overset{GL_n(C(T))}{\sim} v_1$. Hence from the claim it is clear that $(1,0,0) \overset{GL_3(C(S^2))}{\sim} (\mathbf{x},\mathbf{y},\mathbf{z})$ *i.e.* $(\mathbf{x},\mathbf{y},\mathbf{z})$ is completable which is not possible. Hence S^2 is not contractible. $\square$

In Section 5.2, we have seen that if A is the co-ordinate ring of a real algebraic variety and $\vec{a} \in A^n$ is unimodular, then $\vec{a}$ gives rise to a continuous function $F_{\vec{a}} : V_I(\mathbb{R}) \longrightarrow \mathbb{R}^n - \{0\}$. Also if $\vec{b} \in A^n$ is unimodular and $\vec{a} \overset{E_n(A)}{\sim} \vec{b}$, then the corresponding maps $F_{\vec{a}}, F_{\vec{b}} : V_I(\mathbb{R}) \longrightarrow \mathbb{R}^n - \{0\}$ are homotopic.

We shall now investigate the extent to which the converse is valid. Simha's proof ([7]) shows that if T is a compact topological space and $G_{\vec{a}}, G_{\vec{b}} : V_I(\mathbb{R}) \longrightarrow S^{n-1}$ are continuous maps which are homotopic, then the corresponding unimodular rows $\vec{a}$ & $\vec{b}$ satisfy $\vec{a} \overset{GL_n(C(T))}{\sim} \vec{b}$. To investigate the converse, we need a lemma of Vaserstein:-

Lemma 3.1 ([8]). *Let A be a ring and $\vec{a} = (a_1, a_2, \cdots, a_n) \in A^n$ be a unimodular row. Let $\vec{b} = (b_1, b_2, \cdots, b_n)$ and $\vec{c} = (c_1, c_2, \cdots, c_n)$ be such that $\sum_{i=1}^n a_i b_i = \sum_{i=1}^n a_i c_i = 1$. Then there exists a matrix $\alpha \in SL_n(A)$ which can be connected to the identity matrix such that $\vec{b} \overset{\alpha}{\sim} \vec{c}$, if $n \geq 3$.*

Underlying topological idea: If A is the co-ordinate ring of a real algebraic variety and $\vec{a}, \vec{b}, \vec{c} \in A^n$ are unimodular rows satisfying the property $\vec{a}\vec{b}^t = \vec{a}\vec{c}^t = 1$. Then we have two continuous maps $F_{\vec{b}}, F_{\vec{c}} : V_I(\mathbb{R}) \longrightarrow \mathbb{R}^n - \{0\}$.
Claim: $F_{\vec{b}}$ & $F_{\vec{c}}$ are homotopic.
Since $\vec{a}\vec{b}^t = \vec{a}\vec{c}^t = 1$, for each point $p \in V_I(\mathbb{R})$, the vectors $F_{\vec{b}}(p)$ and $F_{\vec{c}}(p)$ are not in antipodal directions (if $F_{\vec{b}}(p) = -F_{\vec{c}}(p)$ means $\langle \vec{a}, \vec{b} \rangle = 1 = -\langle \vec{a}, \vec{c} \rangle$ which is not true). Therefore $H = (1-t)F_{\vec{b}} + tF_{\vec{c}}$ is a continuous map from $V_I(\mathbb{R}) \times \mathcal{I}$ to $\mathbb{R}^n - \{0\}$, which is a homotopy between $F_{\vec{b}}$ & $F_{\vec{c}}$. Thus the claim is proved.

To give algebraic proof of Vaserstein's lemma one needs to compute the determinant of matrices of the kind $I_n + \alpha$, where α is a $n \times n$ matrix of rank 1. Note that a general $n \times n$ matrix of rank ≤ 1 over a field looks like $x^t y$, where $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$. In particular a 2×2 matrix of rank ≤ 1 over a field looks like $x^t y = \begin{pmatrix} x_1 y_1 & x_1 y_2 \\ x_2 y_1 & x_2 y_2 \end{pmatrix}$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

For simplicity we state and prove the lemma in the 2×2 case (the general case being similar).

Lemma 3.2.

$$det \begin{pmatrix} 1 + x_1 y_1 & x_1 y_2 \\ x_2 y_1 & 1 + x_2 y_2 \end{pmatrix} = 1 + x_1 y_1 + x_2 y_2.$$

i.e. $det(I_n + x^t y) = 1 + x y^t$

Proof. Note that

$$det \begin{pmatrix} 1 + x_1 y_1 & x_1 y_2 & 0 \\ x_2 y_1 & 1 + x_2 y_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = det \begin{pmatrix} 1 + x_1 y_1 & x_1 y_2 & x_1 \\ x_2 y_1 & 1 + x_2 y_2 & x_2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= det \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & x_2 \\ -y_1 & -y_2 & 1 \end{pmatrix} = det \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 + x_1 y_1 + x_2 y_2 \end{pmatrix} = 1 + x_1 y_1 + x_2 y_2.$$

Since

$$det\begin{pmatrix} 1+x_1y_1 & x_1y_2 \\ x_2y_1 & 1+x_2y_2 \end{pmatrix} = det\begin{pmatrix} 1+x_1y_1 & x_1y_2 & 0 \\ x_2y_1 & 1+x_2y_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This completes proof of the lemma. $\square$

Remark 3.2. (1) The elementary matrices $E_{ij}(\lambda)$ are of the form $I_n + \alpha$, where α has rank 1.

(2) The matrix $E_{ij}(1) = I_n + e_i^t e_j, i \neq j$. If we conjugate $E_{ij}(1)$ by a matrix $\alpha \in GL_n(k), k$ a field, we obtain the matrix $\alpha E_{ij}(1)\alpha^{-1} = I_n + v^t w$, where $v^t = \alpha e_i^t, w = e_j\alpha^{-1}$. Since $e_i e_j^t = 0, vw^t = 0$ and by Lemma 3.2, it follows that the determinant of the matrix $I_n + v^t w$ is 1.

(3) Let $\sigma : k^n \longrightarrow k$ be a linear transformation and $w \in k^n$ is a non-zero vector such that $\sigma(w) \neq 0$. Define $\sigma' : k^n \longrightarrow k^n$ by $\sigma'(v) = v + \sigma(v)w$. Then σ' is a linear transformation and its matrix is of the form $I_n + \alpha$, where α has rank 1. Since dim $Ker(\sigma) = n-1$, we can choose a basis $x_1, x_2, \cdots, x_{n-1}$ of $Ker(\sigma)$ and therefore $x_1, x_2, \cdots, x_{n-1}, w$ is a basis of k^n. Thus the determinant of σ' with respect to the above basis is seen to be $1 + \sigma(w)$. In particular we can compute the determinant of the reflection transformation which sends v to $v - 2\dfrac{\langle v, w\rangle}{\langle w, w\rangle}w$. Note that in this case $\sigma(v) = -2\dfrac{\langle v, w\rangle}{\langle w, w\rangle}$, in particular $\sigma(w) = -2$. The determinant of the reflection transformation is -1.

(4) If $\sigma(w) = 0$, then we have a basis $v_1, v_2, \cdots, v_{n-1}$ with $v_1 = w$ of $Ker(\sigma)$. Let $v_n \in k^n$ but $v_n \notin Ker(\sigma)$. The determinant of the matrix corresponding to σ' with respect to the basis $v_1, \cdots, v_{n-1}, v_n$ is 1. In particular, it follows as we have seen before that the $det(I_n + v^t w) = 1$, where $wv^t = 0$.

Now we prove Lemma 3.1 of Vaserstein.

Proof. Suppose $\vec{a} \in A^n$ is unimodular and $\vec{b}, \vec{c} \in A^n$ are such that $\vec{a}\vec{b}^t = \vec{a}\vec{c}^t = 1$. Then $\vec{a}(\vec{c} - \vec{b})^t = 0$. So we have a matrix $\alpha = I_n + (\vec{c} - \vec{b})^t\vec{a}$ satisfying the property that $det(\alpha) = 1$ and $\alpha\vec{b}^t = \vec{b}^t + (\vec{c} - \vec{b})^t = \vec{c}^t$. Take $\beta = I_n + (\vec{c} - \vec{b})^t\vec{a}X \in SL_n(A[X])$. Then $\beta(0) = I_n, \beta(1) = \alpha$ proving the lemma. $\square$

Theorem 3.3. *Let $H : T \times \mathcal{I} \longrightarrow S^{n-1}$ be a continuous homotopy such that v_0 and v_1 are the corresponding unimodular rows given by maps $H(x, 0) : T \longrightarrow S^{n-1}$ and $H(x, 1) : T \longrightarrow S^{n-1}$, respectively. Then there exists a matrix $\alpha \in SL_n(C(T))$ such that α can be connected to the identity and $v_0 \overset{\alpha}{\sim} v_1$.*

Proof. Since $\mathcal{I}$ is a compact and connected space, it suffices to prove that if $H(x, t) : T \longrightarrow S^{n-1}$ is a continuous map for sufficiently small t, then $v_0 \overset{\alpha}{\sim} v_t$, where $\alpha \in SL_n(C(T))$ and α can be connected to the identity matrix.
We choose small t so that $v_0(p) + v_t(p) \neq 0$ for every $p \in T$. Define $W : T \longrightarrow S^{n-1}$ by $W(p) = \dfrac{v_0(p) + v_t(p)}{1 + v_0(p)v_t(p)}$. Since $v_0(p)$ and $v_t(p)$ are not antipodal, $v_0(p)v_t(p) \neq -1$. Since any continuous map $T \longrightarrow S^{n-1}$ gives rise to a unimodular row over $C(T)$, we have $v_0(p)W(p) = \dfrac{v_0(p)v_0(p) + v_0(p)v_t(p)}{1 + v_0(p)v_t(p)} = \dfrac{1 + v_0(p)v_t(p)}{1 + v_0(p)v_t(p)} = 1$. Similarly $v_t(p)\ .W(p) = 1$. By Vaserstein's Lemma, there exists $\alpha \in SL_n(C(T))$ such that α can be connected to the identity matrix and $v_0 \overset{\alpha}{\sim} v_t$. $\square$

Note:Consider quaternion algebra
$Q = \{x_1 + ix_2 + jx_3 + kx_4 \mid x_1, x_2, x_3, x_4 \in \mathbb{R}\}$ over $\mathbb{R}$. Let $q_1 = x_1 + ix_2 + jx_3 + kx_4$.

Define $\varphi : Q \longrightarrow Q$ by $T(q) = q_1 q$. Then φ is a linear transformation. Also $\varphi(1) = x_1 + ix_2 + jx_3 + kx_4$, $\varphi(i) = ix_1 - x_2 - kx_3 + jx_4$, $\varphi(j) = jx_1 + kx_2 - x_3 - ix_4$ and $\varphi(k) = kx_1 - jx_2 + ix_3 - x_4$. Therefore

$$matrix(\varphi) = \begin{pmatrix} x_1 & -x_2 & -x_3 & -x_4 \\ x_2 & x_1 & -x_4 & x_3 \\ x_3 & x_4 & x_1 & -x_2 \\ x_4 & -x_3 & x_2 & x_1 \end{pmatrix}.$$

In particular, if $q_1 = ix_2 + jx_3 + kx_4$, then

$$matrix(\varphi) = \begin{pmatrix} 0 & -x_2 & -x_3 & -x_4 \\ x_2 & 0 & -x_4 & x_3 \\ x_3 & x_4 & 0 & -x_2 \\ x_4 & -x_3 & x_2 & 0 \end{pmatrix},$$

which is a skew symmetric matrix. Hence we have a map $\phi' : \mathbb{R}^3 \longrightarrow S'$, where S' is the set of all 4×4 skew symmetric matrices over $\mathbb{R}$ defined by

$$\phi'(x, y, z) = \begin{pmatrix} 0 & -x & -y & -z \\ x & 0 & -z & y \\ y & z & 0 & -x \\ z & -y & x & 0 \end{pmatrix}.$$

Also we have a map $\phi = \phi'_{|S^2} : S^2 \longrightarrow S$, where S is the set of all 4×4 skew symmetric matrices of determinant 1 defined by

$$\phi(x, y, z) = \begin{pmatrix} 0 & -x & -y & -z \\ x & 0 & -z & y \\ y & z & 0 & -x \\ z & -y & x & 0 \end{pmatrix}.$$

Consider $A = \mathbb{R}[X_1, X_2, \cdots, X_n]/I$, where I is an ideal of real algebraic variety in $\mathbb{R}[X_1, X_2, \cdots, X_n]$. Suppose (a_1, a_2, a_3) and (b_1, b_2, b_3) are two unimodular rows over A such that $(a_1, a_2, a_3) \overset{E_3(A)}{\sim} (b_1, b_2, b_3)$. Then the corresponding maps $G_{\vec{a}} : V(\mathbb{R}) \longrightarrow S^2$ and $G_{\vec{b}} : V(\mathbb{R}) \longrightarrow S^2$ are homotopic. Therefore maps $V(\mathbb{R}) \xrightarrow{G_{\vec{a}}} S^2 \xrightarrow{\phi} Q$ and $V(\mathbb{R}) \xrightarrow{G_{\vec{b}}} S^2 \xrightarrow{\phi} Q$ are also homotopic.

Let A be a commutative ring with identity. Suppose $\vec{a} = (a_1, a_2, a_3)$ is a unimodular row, then there exists $\vec{b} = (b_1, b_2, b_3) \in A^3$ such that $\langle \vec{a}, \vec{b} \rangle = \vec{a}\vec{b}^t = \sum_{i=1}^{3} a_i b_i = 1$. Therefore we have a skew symmetric matrix

$$V(\vec{a}, \vec{b}) = \begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & b_3 & -b_2 \\ -a_2 & -b_3 & 0 & b_1 \\ -a_3 & b_2 & -b_1 & 0 \end{pmatrix}$$

.

Let $S =$ Set of all 4×4 skew symmetric matrices over A. Define a relation $\sim$ on S as $\alpha_1 \sim \alpha_2$ if $\alpha_2 = \beta^t \alpha_1 \beta$ for some $\beta \in E_4(A)$. Clearly this is an equivalence relation on S. On the other hand $\overset{E_3(A)}{\sim}$ is an equivalence relation on $Um_3(A)$.

We define a relation $\Psi : Um_3(A)/E_3(A) \longrightarrow S/\sim$ by $\Psi(\vec{a}) = V(\vec{a}, \vec{b})$.

Claim: Ψ is a well defined map.

Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{a'} = (a'_1, a'_2, a'_3)$ be such that $\vec{a'} = \vec{a}\tau$, for some $\tau \in E_3(A)$

Suppose $\vec{a}\vec{b}^t = \sum_{i=1}^{3} a_i b_i = 1$. Then

$$V(\vec{a}, \vec{b}) = \begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & b_3 & -b_2 \\ -a_2 & -b_3 & 0 & b_1 \\ -a_3 & b_2 & -b_1 & 0 \end{pmatrix}.$$

Since $\vec{a}\vec{b}^t = 1$, $\vec{a}\tau\tau^{-1}\vec{b}^t = 1$. This implies that $\vec{a'}\vec{b'}^t = 1$, where $\vec{b'} = \vec{b}(\tau^{-1})^t$. We know that $\tau = \prod_{i=1}^{r} E_{ij}(\lambda)$, where $E_{ij}(\lambda), i \neq j$ is the 3×3 matrix in $SL_3(A)$ which has 1 as its diagonal entries and λ as its $(i,j)^{th}$ entry. So it suffices to prove the claim when

$$\tau = \begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\tau^{-1} = \begin{pmatrix} 1 & -\lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore

$$V(\vec{a'}, \vec{b'}) = \begin{pmatrix} 0 & a_1 & a_1\lambda + a_2 & a_3 \\ -a_1 & 0 & b_3 & -b_2 \\ -a_1\lambda - a_2 & -b_3 & 0 & b_1 - \lambda b_2 \\ -a_3 & b_2 & -b_1 + \lambda b_2 & 0 \end{pmatrix}.$$

Let

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & \tau^t \end{pmatrix}.$$

Then

$$\beta^t V(\vec{a}, \vec{b}) = \begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & b_3 & -b_2 \\ -a_1\lambda - a_2 & -b_3 & b_3\lambda & a_1 - \lambda b_2 \\ -a_3 & b_2 & -b_1 & 0 \end{pmatrix}.$$

Hence

$$\beta^t V(\vec{a}, \vec{b})\beta = \begin{pmatrix} 0 & a_1 & a_1\lambda + a_2 & a_3 \\ -a_1 & 0 & b_3 & -b_2 \\ -a_1\lambda - a_2 & -b_3 & 0 & b_1 - \lambda b_2 \\ -a_3 & b_2 & -b_1 + \lambda b_2 & 0 \end{pmatrix} = V(\vec{a'}, \vec{b'}).$$

Thus Ψ is well defined.

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